

Maintaining Hedonistic Lifestyle from Trading: Optimal Portfolio Strategy for Consumption with Binomial Trees

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Abstract—By this study, we want to provide tools for an economic agent who wants to maintain her maximized intertemporal consumption rate over time period regularly by trading activity in capital market. She would use the diversification principle to compose portfolio that is consisted of risky and risk free assets, with initial wealth by using principle of dynamic programming in stochastic control problem. While our target function is to maximize the discounted utility function over consumption rate, the agent may use this portion of wealth for hedonistic lifestyle without thinking the bequest aim. Motivated by observation that seminal work of Samuelson (1969)'s optimal investment on risky asset was independent of time and wealth process, we were replacing the source of uncertainty in the risky asset by binomial process in order to improve Samuelson's solution. The risky asset price evolution is determined to move in two possible directions of up and down in the next time instantly with certain probability, following binomial trees scheme. Working within this setting, we manage to obtain trading strategies as explicit solution for the optimal investment and consumption decisions which were being expressed as time-dependent functions as well as wealth process feedback form. This solution goes along with inter-temporal concept that our consumption and investment strategies tomorrow will be affected with our wealth today. We exemplify the results by numerical examples and perform Monte Carlo simulation.

Keywords—Dynamic programming, Hamilton-Jacobi-Bellman equation, intertemporal consumption, stochastic optimal control.

I. INTRODUCTION

FORMULATION of optimal investment decision has long been discussed by many scholars in literature. Breakthrough in this research came along when Markowitz [21] introduced the static one period portfolio optimization based on mean-variance as investor's risk preference. Due to the static nature of the problem, optimal solution to the Markowitz' investment problem was independent of time and was not get connected to the changes in wealth process of investor. Samuelson [29] extended Markowitz's work in dynamic way by looking at optimal investment and consumption decisions that maximize the expected total discounted utility of lifetime

consumption in discrete-time framework. The work had assumed that the risk-free asset and risky asset were having continuous distribution and the problem was later resolved by employing standard dynamic programming. However, the investment decision was independent of time and wealth process. Merton ([23],[24]) later extended Samuelson's model in continuous-time by improving underlying uncertainty in the risky asset governed by Ito processes. Working under the hyperbolic absolute risk aversion (HARA) utility function, Merton was able to represent the optimal value function of the portfolio in a "separable" form and obtained explicit solution of the corresponding Hamilton-Jacobi-Bellman equation derived using same dynamic programming argument. The optimal investment-consumption solutions were not presence as time-dependent and wealth feedback form when applied to the power utility or log-utility function. Meanwhile Friend and Blumme [7] shown that most our investor behavior mimic log-utility function toward risky asset demand; and thus we want to bring this problem for this type of investor. Moreover we believe that the dependence of trading and consumption strategies toward wealth process somehow inseparable and by this study, we were trying to tackle within our setting.

In continuous-time framework, improvement to the Merton's model is made by incorporating stochastic economic parameters. They are to be mentioned: Zariphopoulou [31] restricted the maximum risky asset buying with stochastic borrowing constraint, Kim & Omberg [16] adopted stochastic risk premium in the design of non-myopic portfolio, Fleming & Pang [6] added ergodic markov interest rate into their consideration, Lim & Zhou [18] exercised the portfolio with stochastic interest rate, appreciation rate and volatility variables, Brennan & Xia [2002] studied long term investment hence stochastic inflation rates to be highlighted, Bajeux-Besnainou, et al. [2] and Menocin & Vigna [22] utilized Ornstein-Uhlenbeck process for modeling interest rate for different purposes.

While above scholars probed the continuous time framework setting, we would like to exploit the discrete time framework which gives more realistic toward investor observation and portfolio weight adjustment. Another thought,

stock price evolution volatility estimation within continuous framework which is modeled by geometric Brownian motion (GBM) cannot be applied if the stock dynamics was aberrant from GBM. This is where the discrete analysis could tackle the above problems [19].

Some scholar in the old days had developed discrete framework for active investment management analysis in their portfolio in empirical experimental. They are: Mossin [25], Hakanson [13], Leland [17], Ross [28], Huberman & Ross [14] and Grauer & Hakansson [8-10]. Grauer Hakansson [11] improved their empirical work with inflation effect. In contrast, our work is deductive research and along with ours is Grauer & Hakansson [12] which employed Mean Variance approximation to replicate dynamic programming in quadratic utility for optimal terminal wealth purposes, and this model was fitted under partial condition. But what distinguished is we choose dynamic programming over mean variance for active portfolio adjustment reason during investment period.

Within this setting, stock price movement is following binomial tree which moved up and down with some certain probability. It was Sharpe [30] who first had developed this binomial approach to analyze option pricing. Later on, Cox, Ross & Rubinstein [3] proved that within binomial approach, convergence can be reached but in weakly form to log-normal stock price evolution as long as there was no jump process. Moreover, Cox & Rubinstein [4] established the funnel between continuous frame work analyses with discrete if the binomial trees were taken into consideration in stock price evolution. Henceforth, binomial approach can resolve the problem in continuous time framework when the Partial Derivative Equation (PDE) cannot be inferred explicitly, i.e. Cox & Rubinstein [4] puzzled out the option pricing with non-standardized Black Scholes model, where the PDE was awful to be solved. So basically, with modeling through binomial approach, one can substitute of PDE conundrum solving in continuous time. Interested reader may consult to Cox, Ross & Rubinstein [3], Hull [15] for approximations those widely used in practice such as in Options, and other derivatives pricing where the efficacy could be proved. By some transformation, Nelson & Ramaswamy [26] improved the above work so that Sharpe's binomial approach had tractability in computation. Their work will ease us to construct log-normal stock price evolution model within our optimal portfolio.

Hence in this study, we wanted to infuse the discrete optimization of portfolio with the convergent and computation-tractable result of binomial trees transformation in stock price evolution, where number of nodes in tree grows linearly within investment period. Just like in Samuelson and Merton's model, we assumed the existence of one risk-free asset (bond) paying constant interest rate and one risky asset (stock) on both of which trading took place. We also specify necessary conditions for prohibition of short selling in each asset and sufficiency condition to guarantee solvency; that the wealth as well as consumption must be staying positive. Our portfolio is dedicated for investors who want regular

consumption/hedonistic purposes. For one who wants to satisfy consumption as well as bequest may consult to Rizal et al. [27]

Our result was improvement of Samuelson [29] and Merton ([23])'s seminal works: the trading and consumption strategies presented to be wealth and time dependent. This is well suited with intertemporal concept: that our consumption and investment decision today affecting our wealth tomorrow. The paper is organized as follows. Section 2 discusses the two-asset market model in which trading takes place between risk-free bond and stock and the derivation of portfolio dynamics of investor's wealth process using imposed budget constraint. Section 3 discusses the optimal asset allocation and intertemporal consumption model and presents the main results and proofs. Section 4 presents numerical examples and Monte Carlo simulation by which sample paths of stock price process are generated by using coin-toss simulation as can be seen on **Figure 1** below. Section 5 concludes the paper.

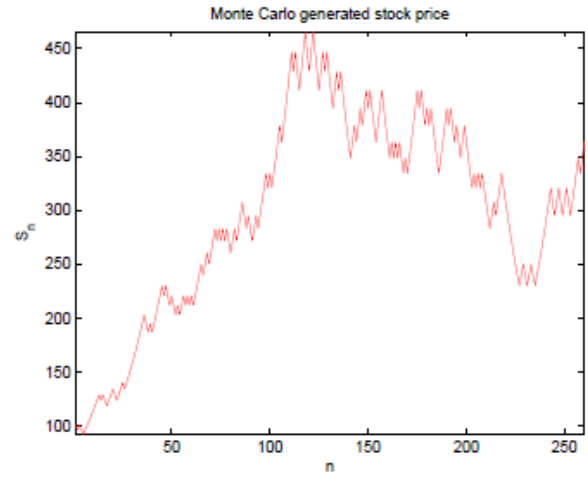


Figure 1 Coin-toss Monte Carlo simulation of stock price model (2.1)-(2.5) with initial price $P_2(0)=100$, expected growth rate $\nu = 15\%$, volatility $\sigma = 30\%$ and period length $\Delta t = 1/52$

II. THE TWO-ASSET MARKET MODEL AND PORTFOLIO DYNAMICS

In the world that investor is interested in maximizing her life time consumption during finite-time period $[0, T]$, she was investing in the two assets portfolio, consisted of bond and stock. Her objective is to find optimal investment and consumption strategy in the way in which she manages to live during that period of time. To solve this problem, we divide the time-period $[0, T]$ into $N \in \mathbb{N}$ equidistant subintervals $[t_n, t_{n+1}]$ with period length Δt , i.e., $t_n = n\Delta t, n = 0, 1, \dots, N$ and $t_N = T$. The price process $B_n := B(t_n)$ of bond grows at a constant interest rate r , i.e., $B_{n+1} = (1 + r\Delta t)B_n$, whereas the price process $S_n := S(t_n)$ of stock evolves according to Binomial process given by equation (2.1) below:

$$S_{n+1} = \begin{cases} u \times S_n, & \text{with probability } (p) \\ d \times S_n, & \text{with probability } (1 - p) \end{cases} \quad (2.1)$$

with $u \geq 1 \geq d$. The probability p and $(1-p)$ are often referred in literature to as the "true" or physical probability. Because the nature of the binomial model (2.1) is multiplicative, the stock price will never become negative. The proposition below gives the law of the stock price.

Proposition 2.1 *The law of the stock price process (2.1) is given by (2.2)*

$$S_n = S_0 \exp\left[\left(p \log\left(\frac{u}{d}\right) + \log(d)\right)n + \log\left(\frac{u}{d}\right)W_n\right], \text{ as } n \rightarrow \infty \quad (2.2)$$

where W_n is normally distributed random process with zero mean and variance $np(1-p)$.

Proof To start with, let us define the log-price $Y_n := \log(S_n)$, $\bar{u} := \log(u)$ and $\bar{d} := \log(d)$. Next, define by Z_k , $k = 1, 2, \dots$ independent and identically distributed (IID) binomial process:

$$\mathbb{P}(Z_k = \bar{u}) = p, \quad \mathbb{P}(Z_k = \bar{d}) = 1 - p, \text{ for all } k.$$

Subsequently, the log-price Y_n can be represented in terms of Z_k as $Y_n = Y_0 + \sum_{k=1}^n Z_k$. For the ease of Monte Carlo simulation, it is more convenient to work with binomial process I_k having outcomes 0 and 1 than with Z_k . For this purpose, we do the following transformation described on following equation (2.3):

$$I_k = \frac{Z_k - \bar{d}}{\bar{u} - \bar{d}} = \begin{cases} 1, & \text{if } Z_k = \bar{u} \\ 0, & \text{if } Z_k = \bar{d} \end{cases} \quad (2.3)$$

or equivalently $Z_k = \bar{d} + (\bar{u} - \bar{d})I_k$. Hence, the binomial process Y_n can be presented as

$$Y_n = Y_0 + \bar{d}n + (\bar{u} - \bar{d})W_n, \quad \text{with } W_n = \sum_{k=1}^n I_k$$

Since I_k , $k = 1, 2, \dots$ are independent and identically distributed, the Law of Large Number says that the sample average $A_n = n^{-1} \sum_{k=1}^n I_k$ converges in probability and almost surely to the expected value $\mu = \mathbb{E}(I_k) = p$ as $n \rightarrow \infty$. Also, the Central Limit Theorem states that for large enough n $\sqrt{n}(A_n - \mu)$ has approximated normal distribution with mean zero and variance $p(1-p)$ implying that W_n is normally distributed with mean np and variance $np(1-p)$.

Figure 2 illustrates distribution of sample paths of log-price change S_n/S_0 generated from binomial model (2.1) using coin-toss Monte Carlo simulation for $t_n = 100$ and $\Delta t = 1/100$. We observe that histogram of the generated sample paths is close to the normal probability density with mean $(p \log(u/d) + \log(d))\Delta t^{-1}t_n$ and variance $(\log(u/d))^2 p(1-p)\Delta t^{-1}t_n$ as dictated by Proposition 2.1. As the period length Δt gets smaller, the stock price dynamics (2.2) can be represented by continuous-time model of simple geometric Brownian motion in the following equation (2.4);

$$S(t_n) = S(0) \exp(vt_n + \sigma W_{t_n}) \quad (2.4)$$

where W_t is the standard Brownian motion, v is the growth rate of log-return of stock, whereas σ is the volatility. In practice,

when matching the binomial model (2.1) to real stock return (assumed to be governed by GBM (2.4)), the parameters d , u and p are chosen to match the expected return and the variance of the logarithm of the price change. By comparing the two models (2.2) and (2.4), the binomial model parameters u , d and p are chosen such as that

$$\begin{aligned} [p \log(u) + (1-p) \log(d)]\Delta t^{-1} &= v \\ [p(1-p)(\log(u) - \log(d))^2]\Delta t^{-1} &= \sigma^2. \end{aligned}$$

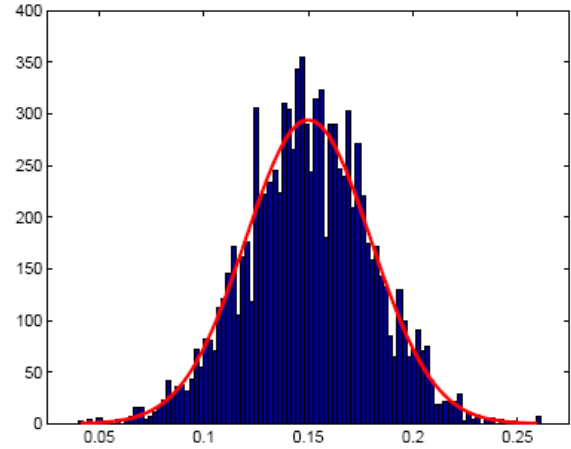


Figure 2 Histogram of log-price S_n/S_0 for binomial model (2.1) generated by performing coin-toss Monte Carlo simulation against normal probability density function with mean $(p \log(u/d) + \log(d))\Delta t^{-1}t_n$ and variance $(\log(u/d))^2 p(1-p)\Delta t^{-1}t_n$, for $t_n = 100$ and $\Delta t = 1/100$. The parameter used are based on the model (2.5) with $v = 0.15\%$ and $\sigma = 0.30\%$

Since there are three unknown parameters to satisfy two equations, there is one degree of freedom in matching the parameter values. To overcome this, it is quite common to set $d = 1/u$. See Luenberger (1998). By doing so, the parameters value of u and p are specified by

$$\begin{aligned} \log(u) &= \sqrt{\sigma^2 \Delta t + (v \Delta t)^2} \\ p &= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{\sqrt{\sigma^2 / (v^2 \Delta t) + 1}} \\ \log(d) &= -\sqrt{\sigma^2 / (v^2 \Delta t)} \end{aligned}$$

For Δt is relatively small, the above can be approximated by equation (2.5)

$$\begin{aligned} u &= e^{\sigma \sqrt{\Delta t}} \\ p &= 1/2 + 1/2 (v/\sigma)\sqrt{\Delta t} \\ d &= e - \sigma \sqrt{\Delta t} \end{aligned} \quad (2.5)$$

Figure 1 illustrates the binomial model (2.1)-(2.5) of stock price movement in weekly periods with an expected yearly growth rate of $v = 15\%$, yearly volatility rate of $\sigma = 30\%$ with weekly period length of $\Delta = 1/52$. The sample paths are generated based on coin-toss Monte Carlo simulation with the head and tail denoting the stock moving up and down, respectively.

To prevent stock return in the above market model from dominating the return on the bond, and vice versa, we impose the following condition on the bond return depicted on (2.6)

$$d \leq (1 + r\Delta t) \leq u. \quad (2.6)$$

The above is sufficient and necessary condition for the absence of arbitrage opportunity in the market model. We refer among others to Proposition 2.3 in Björk [2009] for details. Following Björk [2009], (2.6) is equivalent to saying that $1 + r\Delta t$ is convex combination of u and d , i.e.,

$$1 + r\Delta t = q_u \times u + q_d \times d,$$

where q_u and q_d are martingale probabilities defined by following equation (2.7)

$$q_u = \frac{(1+r\Delta t)-d}{u-d} \text{ and } q_d = \frac{u-(1+r\Delta t)}{u-d} \quad (2.7)$$

Trading takes place between stock and bond. While investing in these assets, investor consumes part of the trading proceeds for her living. We assume that there is no additional exogenous capital injection into the portfolio nor withdrawal of money, except the consumption the investor makes over time; that is to say that the portfolio is assumed to be self-financing. Throughout the rest of this paper we denote by $X_n := X(t_n)$ the investor's total wealth at time t_n , $n = 0, 1, \dots, N$. Furthermore, we define by $\omega_{1,n}$ and $\omega_{2,n}$ the portion of asset that investor holds respectively in bond and stock at time t_n , satisfying the portfolio constraint: $\omega_{1,n} + \omega_{2,n} = 1$ for all $n = 0, 1, \dots, N$. The number of shares in each asset held is given by

$$\theta_{1,n} := \frac{\omega_{1,n} \times n}{B_n} \text{ and } \theta_{2,n} := \frac{\omega_{2,n} \times n}{S_n}$$

We impose that the number of assets held by the investor is assumed to be infinitesimally divisible meaning that the number of shares in traded assets can be fractional number.

Following the portfolio constraint, the investor's wealth at time t_n can be written in terms of the asset prices and the number of shares held in each asset as what follows equation (2.8)

$$X_n = \theta_{1,n}B_{1,n} + \theta_{2,n}S_n = \theta_n^T P_n \text{ for all } n = 0, 1, \dots, N \quad (2.8)$$

where $\theta_n = (\theta_{1,n}, \theta_{2,n})$ and $P_n = (B_n, S_n)^T$. To derive the portfolio dynamics, we firstly derive the portfolio budget constraint dictating the amount of consumption investor can withdraw from the portfolio. To do this, we denote by $C_n := C(t_n)$ the amount of consumption withdrawn during the time period t_n to t_{n+1} . The portfolio budget constraint is given below.

Proposition 2.2 (Budget constraint) *The consumption C satisfies equation (2.9) below*

$$-C_n + \theta_n^T P_{n+1} = \theta_{n+1}^T P_{n+1} \text{ for all } n = 0, 1, \dots, N. \quad (2.9)$$

Proof The investor's wealth value at time t_{n+1} is equal to the value of old portfolio sold at the current price: $\theta_n^T P_n$. The

proceed of this sell is taken away for consumption and the remaining of which is reinvested again in the assets. \square

By incorporating the budget constraint (2.9) into portfolio rebalancing, the investor's wealth process $X = (X_n : n = 0, 1, \dots, N)$ evolves according to the following dynamics.

Proposition 2.3 (Portfolio dynamics) *The investor's portfolio dynamics is given by equation (2.10) below*

$$X_{n+1} = (1 + r\Delta t)X_n + \theta_{2,n}[S_{n+1} - (1 + r\Delta t)S_n] - C_n \quad (2.10)$$

The first term on the right-hand side is the time value of money of investor's wealth held from time t_n to time t_{n+1} ; the second term is the proceeds investor makes from trading on stock in excess of the holding period time value of money of the stock held during that time period. The last term is the amount of consumption withdrawn from the portfolio during the period.

Proof Following (2.8), the increment of wealth during the period t_n and t_{n+1} is

$$X_{n+1} - X_n = \theta_n^T \Delta P_n - C_n,$$

where we have denoted $\Delta P_n := (\Delta B_n, \Delta S_n)^T$ and used (2.9) in the last equality. The proof is established after taking the fact that $\Delta B_n = r\Delta t B_n$ and $\theta_{1,n}B_n = X_n - \theta_{2,n}S_n$.

From view point of time t_n , we observe that the source of uncertainty in the wealth process comes from risky asset. To simplify our notation in the main results below, we define:

$$\tilde{R} := (1 + r\Delta t), \quad \tilde{u} = u - \tilde{R}, \quad \tilde{d} = d - \tilde{R}, \\ \pi_n := \theta_{2,n}S_n$$

Hence, following (2.1) we have that

$$X_{n+1} = \begin{cases} X_{n+1}^{(+)} := \tilde{R}X_n + \pi_n \tilde{u} - C_n, & \text{with probability } p \\ X_{n+1}^{(-)} := \tilde{R}X_n + \pi_n \tilde{d} - C_n, & \text{with probability } (1 - p) \end{cases} \quad (2.11)$$

III. THE OPTIMAL CONSUMPTION-MAXIMIZING PORTFOLIO

Assuming that the underlying uncertainty in the wealth process $X = (X_n : n = 0, 1, \dots, N)$ is generated by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = \{\mathcal{F}_n = \sigma(X_n), n = 0, 1, \dots, N\}$ is satisfying the usual conditions. On this probability space the wealth process X is defined. We require that the control process (π, C) is assumed to be \mathcal{F} -adapted to X . This is to say that the control is only allowed to depend on past observed value of X . One way of looking at this is by allowing the control to be a time-dependent function of X , i.e., $\pi_n = \pi_n(X_n)$ and $C_n = C_n(X_n)$. Furthermore, we denote by \mathbb{E}_n the \mathcal{F}_n -conditional expectation operator associated with the law of X under which X starts at X_n at time t_n .

The investor's objective is to look for optimal investment and consumption strategy (π, C) within the admissible control set $\mathcal{A}_{n,N} := \{(\pi_m, C_m) : m = n, \dots, N\}$ that maximizes the value function J of the portfolio describing the expected total

discounted log-utility of consumption. The value function J of the portfolio is defined by Equation (3.1) below

$$J_n^{(\pi, C)}(t, c) = \mathbb{E}_n \left\{ \sum_{m=n}^{N-1} (1 + r\Delta t)^{-(m-n)} \mathcal{U}_1(C_m) \right\} \quad (3.1)$$

Throughout this paper we use the log-utility function for consumption, i.e., $\mathcal{U}_i(c) = \log(c)$, $i = 1, 2$. The investor is interested in finding the optimal value function

$$V_n(X_n) = \max_{(\pi, C) \in \mathcal{A}_{n,N}} J_n^{(\pi, C)}(X_n) \quad \text{for all } n \quad (3.2)$$

Our goal is to characterize the optimal value function V and determine the optimal asset allocation and consumption strategies (π, C) within the admissible control set \mathcal{A} . Before we present the main results, let us define the following variables. These variables will be used throughout the remaining of this paper based on which the main results are expressed.

$$F = 1 - \frac{p\tilde{u} + (1-p)\tilde{d}}{\tilde{d}}$$

$$G = 1 - \frac{p\tilde{u} + (1-p)\tilde{d}}{\tilde{u}}$$

$$H = 1 - \left(\frac{p\tilde{u} + (1-p)\tilde{d}}{\tilde{u}} \right) \left(\frac{p\tilde{d} + (1-p)\tilde{u}}{\tilde{d}} \right)$$

It is straightforward to check that the condition (2.6) requires these variables to be positive.

Applying the stochastic dynamic programming methodology in discrete-time, optimal solution to the investor's investment and consumption problem (3.1) - (3.2) is outlined below.

Proposition 3.1 (i) *The optimal value function V is given by (3.3) below*

$$V_n(X_n) = \phi_n \log(X_n) + \psi_n, \quad \text{for all } n \quad (3.3)$$

where the functions $\phi_n = \phi(t_n)$ and $\psi_n = \psi(t_n)$ solve the recursive equations:

$$\begin{cases} \phi_n = 1 + \tilde{R}^{-1} \phi_{n+1} \\ \phi_N = 0 \\ \psi_n = \log \left[\frac{\tilde{R}^2 FG}{\tilde{R} FG + \phi_{n+1} H} \right] + p \phi_{n+1} \tilde{R}^{-1} \log \left[\left(\frac{\phi_{n+1} H}{\tilde{R} FG + \phi_{n+1} H} \right) F \tilde{R} \right] \\ \quad + (1-p) \phi_{n+1} \tilde{R}^{-1} \log \left[\left(\frac{\phi_{n+1} H}{\tilde{R} FG + \phi_{n+1} H} \right) \left(\frac{\tilde{u} + (F-1)\tilde{d}}{\tilde{u}} \right) \tilde{R} \right] + \tilde{R}^{-1} \psi_{n+1} \\ \psi_N = 0 \end{cases} \quad (3.4)$$

The recursive equation for ϕ can be worked out to get

$$\phi_n = \frac{\tilde{R}}{\tilde{R}-1} \left(1 - \frac{\tilde{R}^n}{\tilde{R}^N} \right)$$

(ii) The optimal asset allocation and consumption strategies are given in wealth process feedback form to be appeared in equation (3.5) and (3.6)

$$\pi_n^*(X_n) = \left[\frac{(F-1)\tilde{R}}{\tilde{u}} \frac{\phi_{n+1} H}{\tilde{R} FG + \phi_{n+1} H} \right] X_n \quad (3.5)$$

$$C_n^*(X_n) = \left[\frac{\tilde{R}^2 FG}{\tilde{R} FG + \phi_{n+1} H} \right] X_n \quad (3.6)$$

(iii) Prohibition of short selling on stock applies if and only if $p > q_u$.

(iv) Sufficient condition on prohibition of short selling on bond requires that $< \frac{q_u \times u}{(1+R)}$.

(v) Sufficient condition on prohibition of short selling on bond and stock: $q_u < p < \frac{q_u \times u}{(1+R)}$

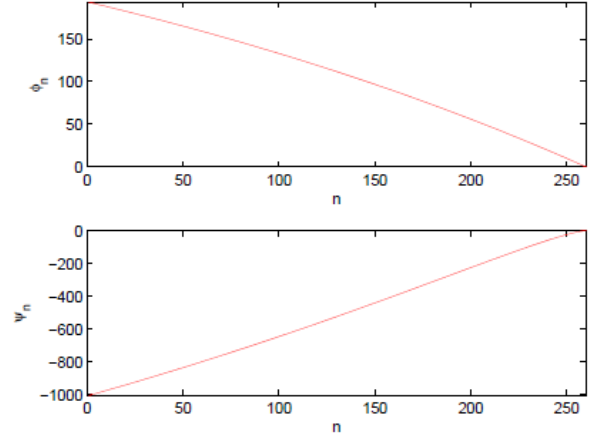


Figure 3 Function $\phi(t)$ and $\psi(t)$ against time t following recursive equation (3.4)

Proof Using the standard stochastic dynamic programming arguments the optimal value function V (3.2) must satisfy equation (3.7) below;

$$V_n(X_n) = \max_{(\pi, C)} \mathbb{E} \{ \mathcal{U}_1(C_n) + \tilde{R}^{-1} V_{n+1}(X_{n+1}) \} \quad \text{with} \\ V_N(x) = 0 \quad \text{for all } x. \quad (3.7)$$

Assume that the value function V is of the form (3.3). Direct substitution to (3.7) gives

$$\phi_n \log(X_n) + \psi_n = \max_{(\pi, C)} \left\{ \log(C_n) + \frac{\psi_{n+1}}{\tilde{R}} + \frac{\phi_{n+1}}{\tilde{R}} [p \log(X_{n+1}^{(+)}) + (1-p) \log(X_{n+1}^{(-)})] \right\}.$$

Applying Euler condition on the static optimization with respect to π gives

$$0 = \frac{p \phi_{n+1}}{X_{n+1}^{(+)}} \tilde{u} + \frac{(1-p) \phi_{n+1}}{X_{n+1}^{(-)}} \tilde{d} \\ = \phi_{n+1} (\tilde{R} X_n - C_n) [p \tilde{u} + (1-p) \tilde{d}] + \pi_n \tilde{u} \tilde{d} \phi_{n+1}.$$

from which we obtain that

$$\pi_n = - \frac{(p \tilde{u} + (1-p) \tilde{d})}{\tilde{u} \tilde{d}} (\tilde{R} X_n - C_n). \quad (3.8)$$

Using the same reasoning through applying Euler condition with respect to C gives

$$0 = \tilde{R} [(\tilde{R} X_n - C_n) + \pi_n \tilde{u}] [(\tilde{R} X_n - C_n) + \pi_n \tilde{d}] \\ - C_n \phi_{n+1} \{ p [(\tilde{R} X_n - C_n) + \pi_n \tilde{d}] + (1-p) [(\tilde{R} X_n - C_n) + \pi_n \tilde{u}] \}.$$

By inserting (3.8) in the last equality above, we have after some algebra that

$$0 = \phi_{n+1}(\tilde{R}X_n - C_n) \left[1 - \left(\frac{p\tilde{u} + (1-p)\tilde{d}}{\tilde{d}} \right) \left(\frac{p\tilde{d} + (1-p)\tilde{u}}{\tilde{u}} \right) \right],$$

which expression of C_n^* in (3.6) follows. The optimal asset allocation decision π_n^* (3.5) is obtained after replacing C_n in (3.8) by C_n^* , which indeed proves the claim (ii). When substituting C_n^* and π_n^* to the wealth process (2.11) we have

$$X_{n+1}^{(+)} = \frac{\phi_{n+1}H}{(\tilde{R}FG + \phi_{n+1}H)} \tilde{R}FX_n$$

$$X_{n+1}^{(-)} = \left(\frac{\tilde{u} + (F-1)\tilde{d}}{\tilde{u}} \right) \frac{\phi_{n+1}H}{(\tilde{R}FG + \phi_{n+1}H)} \tilde{R}FX_n$$

Inserting these in the Bellman's equation above we have after simplification that

$$\begin{aligned} \phi_n \log(X_n) + \psi_n &= \left(1 + \frac{\phi_{n+1}}{\tilde{R}} \right) \log(X_n) + \tilde{R}^{-1} \psi_{n+1} \\ &+ \log \left[\frac{\tilde{R}^2 FG}{(\tilde{R}FG + \phi_{n+1}H)} \right] \\ &+ p \phi_{n+1} \tilde{R}^{-1} \log \left[\left(\frac{\phi_{n+1}H}{(\tilde{R}FG + \phi_{n+1}H)} \right) F \tilde{R} \right] \\ &+ (1-p) \phi_{n+1} \tilde{R}^{-1} \log \left[\left(\frac{\phi_{n+1}H}{(\tilde{R}FG + \phi_{n+1}H)} \right) \left(\frac{\tilde{u} + (F-1)\tilde{d}}{\tilde{u}} \right) \tilde{R} \right] \end{aligned}$$

By choosing the function ψ by the one given in (3.4), we arrive at establishing the claim (i). The proof for claim in (iii)-(v) is complete by checking that the conditions $\omega_{2,n} > 0$, $\omega_{1,n} > 0$ and $0 < \omega_{2,n} < 1$ are satisfied under the imposed condition on P.

IV. NUMERICAL EXAMPLE AND MONTE CARLO SIMULATION

To motivate the main results presented in Proposition 3.1, we discuss some numerical examples and perform Monte Carlo simulation in attempt to mimic practical example for stock price. For this purpose, we assume that a stock price has an expected yearly growth rate of 15% and volatility of 30% with the initial stock price \$100. Furthermore, we assume the existence of risk free bond with annual return of 12.5%. Investor has initial endowment of \$1000 to invest in the two assets. Following Proposition 3.1 the time varying optimal portfolio weights and consumption are depicted in Figure 4 below. We observe that the composition of wealth in the portfolio is initially more dominated by stock as much as 77.68% than the bond comprising of 22.32%. At terminal date T, we notice that the investor holds only the bond 100%. The consumption is increasing from \$5.20 to \$1000 at the period end. The value function of the portfolio is concave and decreasing from \$329.75 to \$0 which is equal to the end value of portfolio wealth.

Next, we attempt to simulate the stock price process from the binomial model (2.1)-(2.5) based on weekly basis (with period length of $\Delta t = 1/52$). The simulation is performed by

conducting coin-toss Monte Carlo simulation; that is whenever the head is up, the stock price goes up accordingly with multiplier u and goes down by d much when the tail comes up. The length of simulation period is for five years in weekly basis. Figure 1 displays the simulation result of stock price process. Using the realization of the sample paths of stock price process S , we desire to simulate the market-to-market sample paths of the wealth process X moves up and down in decreasing manner. This is due to withdrawal of fund over time by the investor for her living consumption C . Such random movements are similarly found for the portfolio time-value process V . In contrary to these observations, we see that the consumption moves up and down in increasing manner. All of these numerical findings displayed in **Figure 5** resemble the theoretical facts presented in Proposition 3.1.

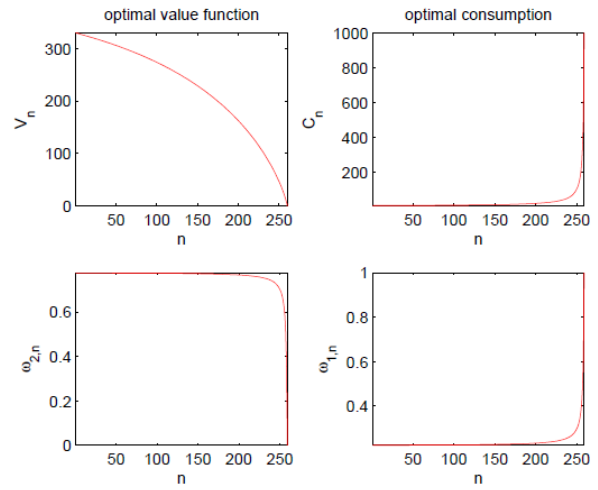


Figure 4 Optimal portfolio weights and consumption based on Proposition 3.1

V. CONCLUSION

Throughout this paper we have seen that the presence of intertemporal consumption in the discrete-time framework enables us to get time-variation effect on the optimal portfolio and consumption strategy. The optimal strategy obtained is given in a closed feedback form in terms of wealth process thanks to the log-utility function used together with the linearity of the portfolio dynamics with respect to the wealth and control process. These findings are in contrast to that of found for the continuous-time counter-part. The binomial approach employed in modeling the stock price movement results in a very fast computation of the optimal solution in terms of a recursive equation. The necessary and sufficient conditions on the prohibition of short selling on stock and bond are verifiable directly from the optimal solution.

The conditions were given explicitly in terms of the physical and martingale probabilities of the stock price movement. For future work, we are looking at is to incorporate transaction costs imposed when rebalancing portfolio over time.

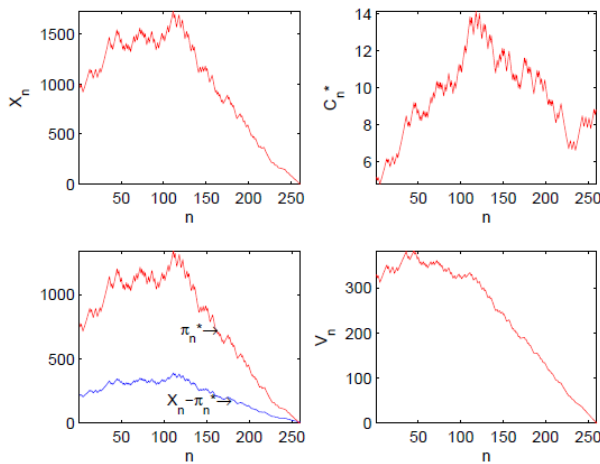


Figure 5 Monte Carlo simulation of wealth process X_n (2.10), consumption C_n (3.6), the amount invested in stock π_n (3.5) and bond $X_n - \pi_n$ and the value process V_n (3.3)

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