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Bi-Dirichlet-type Problems with Polynomial Data in A Unit Sphere in $\mathbb{R}^3$

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Abstract— We studied Bi-Dirichlet boundary value problem of BiLaplace equation. The problem is reformulated as a systems of Laplace-Poisson equation with Dirichlet problems. We utilize an exact algorithms for solving Laplace equations with Dirichlet conditions with polynomial functions data. The algorithm requires differentiation of the boundary function, but no integration.

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I. Introduction

In this paper, we studied Bi-Laplace equation which have Bi-Dirichlet type boundary value problem with polynomials data in $\mathbb{R}^3$. We approached the solution by Lemma that has been formulated by Axler and Ramey (1995) [2] and Lemma by Herzog (2000) [3].

Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, thus for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of non-negative integers, we say x^α as monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$. The degree of x^α is $\alpha_1 + \alpha_2 + \alpha_3$. A polynomial is said to be homogenous of degree m if it is a finite linear combination of monomial x^α of degree m, with $m = 0,1,2,\cdots$. Let $\mathcal{P}_m$ denotes the vector space of polynomials in $\mathbb{R}^3$, homogenous of degree m, and $\mathcal{H}_m$ is the subspace of harmonic polynomials of degree m, then we have property by Axler and Ramey following

$$\mathcal{P}_m = \mathcal{H}_m + |x|^2 \mathcal{P}_{m-2}$$

Theorem 1.1 *If $p \in \mathcal{P}_m$, then $p = \Lambda_m(p) + |x|^2 q$, for some $q \in \mathcal{P}_{m-2}$ and $\Lambda_m(p) \in \mathcal{H}_m$.*

Theorem 1.1 leads to the following corollary, which gives the well-known decomposition (2) and an explicit formula for p_m.

Corollary 1.2 *Every $p \in \mathcal{P}_m$ can be uniquely written in the form*

$$p = p_m + |x|^2 p_{m-2} + \cdots + |x|^{2k} p_{m-2k}, \qquad \textbf{(1)}$$

where $k = \left[\frac{m}{2}\right]$ and $p_j \in \mathcal{H}_j$ for each j. Furthermore, $p_m = \Lambda_m(0)$.

Let Ω is a unit sphere in $\mathbb{R}^3$, with the boundary $\Gamma = \partial\Omega = \{x \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 < 1\}$. The bi-Dirichlet type problem can be formulated as following:

$$\Delta^2 u = 0 \text{ in } \Omega \qquad \textbf{(2)}$$

with boundary value problems

$$u|_\Gamma = p$$

and

$$\Delta u|_\Gamma = q$$

where, p and q is polynomials function.

The main result of this paper is to state the Theorem for the solution of problem in (2). The solution of this problem is a polynomial function that is given in Theorem 3.1.

II. Technical Lemmas

This following lemmas are needed to solve bi-Dirichlet problem:

Lemma 2.1 *If $p \in \mathcal{P}_m$, then the solution to the Dirichlet problem $u \in \mathcal{P}_m$ with boundary data $u|_\Gamma = p$ is*

$$p_m + p_{m-2} + \cdots + p_{m-2k} \qquad \textbf{(3)}$$

where $k = \left[\frac{m}{2}\right]$ and $p_m, p_{m-2}, \cdots, p_{m-2k}$ are the harmonic polynomials.

Proof:

Suppose $p \in \mathcal{P}_m$, because $|x| = 1$ in boundary, by using (1) we have (3) equals to p on Γ. Obviously (3) is harmonic on $\mathbb{R}^3$, and hence its restriction to Ω is the solution to the Dirichlet problem.

Lemma 2.1 said that if we have polynomial data for dirichlet problem then we can obtain the solution of the Laplace equation with this boundary value problem. The solution is a linear combination of harmonic functions. We recommend you to study the algorithm to obtain this harmonic polynomials in [2].

By setting $p_{m-1}, p_{m-3}; \cdots$ are equal to 0, the solution u can be written as

$$u = \sum_{j=0}^{m} p_j$$

where $\sum_{j=0}^{m} p_j \big|_\Gamma = p$.

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Lemma 2.2 [3] *Given* $p \in \mathcal{P}_m$ *and* $q \in \mathcal{P}_{m+2}$ *then there exists* $u \in \mathcal{P}_{m+2}$ *such that*

$$\Delta u(x) = q(x) \quad x \in \Omega \quad and \quad u|_\Gamma = p(x).$$

III. Theorem

Here we provide the Theorem about the solution of bi-Dirichlet type problem. The wxistence of this solution can be presented in the following theorem:

Theorem 3.1 *Given* $p \in \mathcal{P}_m$ *and* $q \in \mathcal{P}_{m+2}$*, then there exists* $u_h \in \mathcal{P}_m$ *and* $u_2 \in \mathcal{P}_{m+2}$ *such that* $u = u_h + u_p$ *is the solution of* **(2)**.

Proof:

The bi-Dirichlet type problem was presented in **(2)** can be written as the following:

$$\Delta u = v; \quad u|_\Gamma = p \qquad \textbf{(4)}$$

and

$$\Delta v = 0; \quad v|_\Gamma = q \qquad \textbf{(5)}$$

The solution of problem **(4)** can be formulated if we had solved the problem **(5)**. The solution of problem **(5)**, $v \in \mathcal{P}_{m+2}$, can be obtained by using Lemma 2.1 which is a harmonic polynomials function.

Suppose that the solution of **(4)** is $u = u_h + u_p$, where u_h is the solution of this problem:

$$\Delta u_h = 0; \quad u_h|_\Gamma = p \qquad \textbf{(6)}$$

and u_p is the solution of this problem:

$$\Delta u_p = v; \quad u_p\big|_\Gamma = 0 \qquad \textbf{(7)}$$

The solution uh can be obtained by using Lemma 2.1 which is the harmonic polynomials function. Next, by solving problem **(7)** when the solution of this problem is guaranteed by Lemma 2.2, so we have up.

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Bi-Dirichlet Problem

Find the biharmonic function on B that equals p_{D} on ∂B whose outward normal derivative on ∂B equals q_N.

A function u is called biharmonic if $\Delta(\Delta u) = 0$. The solution to the bi-Dirichlet problem is the function

$$\frac{|x|^2 - 1}{2}\left(\sum_{j=0}^{M} jp_j - \sum_{j=0}^{m} jp_j\right) + \sum_{j=0}^{m} p_j \qquad \textbf{(8)}$$

A straightforward calculation shows that the Laplacian of this function equals

$$\sum_{j=0}^{M} (n+2j)q_j - \sum_{j=1}^{m} j(n+2j)p_j \qquad \textbf{(9)}$$

which is harmonic, and hence **(8)** is biharmonic. The function **(8)** obviously equals p on ∂B An easy calculation shows that the outward normal derivative on ∂ of **(8)** equals q.